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Subcentral Ideals in Generalized Effect Algebras†

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In this paper, we introduce *subcentral* ideals in the class of cancellative positive partial abelian monoids (CPAMs). Every complementary pair of subcentral ideals in a CPAM $\mathcal{P}$ corresponds to a subdirect decomposition of $\mathcal{P}$. If this decomposition is direct, the corresponding ideals are called *central*. Subcentral ideals are characterized as central elements in the lattice of the recently introduced so-called R_1-ideals. Every subcentral ideal is a central element in the lattice of all ideals. A subcentral ideal I is central iff I is Riesz ideal. In an upper-directed CPAM, every subcentral ideal is central.

1. INTRODUCTION

A cancellative positive partial abelian monoid (CPAM) is an associative, commutative partial groupoid with a neutral element 0 in which the cancellative law holds and 0 is the smallest element in the partial order induced by the binary operation.

The study of partial abelian semigroups started in ref. 11, with further development in refs. 10, 6, and 1. There is a hierarchy of CPAMs: effect algebras, MV-algebras, orthomodular lattices are CPAMs, as well as various types of subalgebras (in particular, ideals) of these structures.

In the present paper, we introduce *subcentral* ideals in the class of CPAMs. An ideal I is called subcentral iff every element can be decomposed in a unique way into I and some other fixed ideal I', which is called a complement of I.

First, we investigate basic properties of subcentral ideals. We show that every subcentral ideal has a unique complement and that for every subcentral ideal I in a CPAM P, there is an idempotent full homomorphism, which maps

† This paper is dedicated to the memory of Prof. Rüttimann.

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$\mathcal{P}$ onto I. Then we prove a one-to-one correspondence between pairs (I, I') of subcentral ideals and certain subdirect decompositions of $\mathcal{P}$. If the corresponding decomposition is direct, we say (in accordance with ref. 5) that I is *central*.

Then we study subcentral ideals in the context of ideal types introduced in refs. 1 and 6. We prove that subcentral ideals are exactly the central elements in the lattice of R_1-ideals. A subcentral ideal I is central iff I is a Riesz ideal. This implies that in the class of upper-directed CPAMs, every subcentral ideal is central. This means that "subcentral ideal" and "central ideal" are two possible generalizations of the notion "central ideal of an effect algebra" for the class of CPAMs.

2. DEFINITIONS

Definition 1. Let $\mathcal{P} = (P, \oplus, 0)$ be a partial algebra with a nullary operation 0 and a binary partial operation $\oplus$. Denote the domain of $\oplus$ by $\perp$. $\mathcal{P}$ is called a *partial abelian monoid* (PAM) iff for all $a, b, c \in P$ the following conditions are satisfied:

(P1) $a \perp b$ implies $b \perp a$, $a \oplus b = b \oplus a$.
(P2) $b \perp c$ and $a \perp b \oplus c$ implies $a \perp b$, $a \oplus b \perp c$, $a \oplus (b \oplus c) = (a \oplus b) \oplus c$.
(P3) $a \perp 0$ and $a \oplus 0 = a$.

A partial abelian monoid is called a *cancellative positive PAM* (*CPAM*) iff the following conditions are satisfied.

(P4) $a \oplus b = a \oplus c$ implies $b = c$.
(P5) $a \oplus b = 0$ implies $a = 0$.

In a CPAM $\mathcal{P}$, we denote $a \leq b$ iff $a \oplus c = b$ for some $c \in P$. We denote $c = a \ominus b$ iff $a = b \oplus c$. It is easy to prove that $\ominus$ is a well-defined partial operation. A CPAM $\mathcal{P}$ is called *upper-directed* iff for every $a,b \in P$, there exists $c \in P$, such that $a \leq c$ and $b \leq c$. *Effect algebra* [23] is an upper-bounded CPAM. The greatest element in an effect algebra is denoted by 1. In ref. 8 a class of partial algebras named *D-posets* was introduced. D-posets are essentially equivalent to effect algebras, but they are defined with $\ominus$ as a primary operation. A class of structures equivalent to CPAMs but based on $\ominus$ was introduced in ref. 7—*abelian RI-posets*.

Definition 2[10, 6]. Let $\mathcal{P}$ be a PAM. Let $\sim$ be a relation on P such that:

(C1) $\sim$ is an equivalence relation.
(C2) $a_1 \sim a_2$, $b_1 \sim b_2$, $a_1 \perp b_1$, $a_2 \perp b_2$ imply $a_1 \oplus b_1 \sim a_2 \oplus b_2$.

Then $\sim$ is called a *weak congruence* of $\mathcal{P}$. If, in addition,

(C3) $a \sim b$, $a \perp c$ imply there is $c_1 \sim c$, such that $b \perp c_1$ then $\sim$ is called a *congruence* of $\mathcal{P}$. If, in addition,

(C4) $a \perp b$, $a_1 \perp b_1$, $a_1 \sim a$, $a_1 \oplus b_1 \sim a \oplus b$ imply $b_1 \sim b$, then $\sim$ is called a *c-congruence*.

In ref. 10 it was proven that for a congruence $\sim$ on a PAM $\mathcal{P}$, the quotient $P/\sim$ is a PAM. The following condition was considered in ref. 1:

(C5) $a \sim b \oplus c$ iff $\exists a_1, a_2$ with $a_1 \sim b$, $a_2 \sim c$, and $a = a_1 \oplus a_2$

If $\sim$ is a relation of a PAM P satisfying (C1), (C2), and (C5), then the quotient is a PAM [1].

Definition 3. An *order ideal* of a CPAM is a nonempty subset I of P having the property: If $x \oplus y \in I$, then $x, y \in I$. If, in addition, $x \perp y$, $x, y \in I$, implies $x \oplus y$ in I, then I is called an *ideal* of P.

We denote the system of all ideals of a CPAM $\mathcal{P}$ by $I(P)$. $(I(P), \subseteq)$ is a complete lattice with smallest element $\{0\}$ and greatest element P. The meet of two arbitrary ideals is their intersection. The join has more complex structure.

Having an ideal I of a CPAM $\mathcal{P}$, the relation $\sim_I$ on P is defined as follows: $a \sim_I b$ iff there are $a_1, b_1 \in I$ such that $a_1 \le a$, $b_1 \le b$, $(a \ominus a_1) = (b \ominus b_1)$. We say that an ideal I is *weakly algebraic* iff $\sim_I$ satisfies the (C1) and (C2) properties.

For a weakly algebraic ideal I we denote $P/\sim_I$ briefly by P/I and the equivalence class of a under $\sim_I$ by $[a]_I$.

Definition 4 [4]. Let $\mathcal{P} = (P, \oplus, 0)$ be a CPAM, and let $A \subseteq P$, $0 \in A$. Define $\oplus_A$ on A as follows. $a \oplus_A b$ is defined and equals c if and only if $a \oplus b = c$ in $\mathcal{P}$ and $c \in A$. Then $(A, \oplus_A, 0)$ is called a *relative subalgebra* of $\mathcal{P}$.

The following example shows that there are relative subalgebras of CPAMs which are not CPAMs.

Example 1. It is trivial that the set $\langle 0, \infty)$ of nonnegative real numbers, equipped with the usual $+$ operation, is a CPAM. Let $A = \{0, 1/4, 1/2, 1\}$. Then the relative subalgebra of $\langle 0, \infty)$ associated with A fails to satisfy the (P2) condition: $(\frac{1}{4} + \frac{1}{4}) + \frac{1}{2}$ exists, but $\frac{1}{4} \not\perp \frac{1}{2}$.

Proposition 1. Let $\mathcal{P}$ be a CPAM. Let A be an order ideal in $\mathcal{P}$. Then the relative subalgebra associated with A is a CPAM.

Proof. Obviously, the conditions (P1), (P3), (P4), (P5) hold. Assume $b \perp_A c$, $a \perp_A b \oplus_A c$. Then $a \oplus (b \oplus c) \in A$, which implies $a \oplus b \in A$, i.e., $a \perp_A b$. Since $(a \oplus b) \oplus c = a \oplus (b \oplus c) \in A$, we see that $a \oplus b \perp_A c$. ■

Definition 5 [10]. Let $\mathcal{P}_1$ and $\mathcal{P}_2$ be CPAMs. Let $\phi : P_1 \mapsto P_2$ be such that

(H1) $x \perp y$ implies that $\phi(x) \perp \phi(y)$ and $\phi(x \oplus y) = \phi(x) \oplus \phi(y)$.

Then ϕ is called a *homomorphism* of CPAMs. If, in addition,

(H2) $\phi(x) \perp \phi(y)$ and $\phi(x) \oplus \phi(y) \in \phi(P_1)$ imply that there are x_1, $y_1 \in P_1$ such that $\phi(x_1) = \phi(x)$, $\phi(y_1) = \phi(y)$ and $x_1 \perp y_1$

then ϕ is called a *full homomorphism*, of CPAMs. A bijective full homomorphism is called an *isomorphism*.

3. SUBCENTRAL IDEALS AND RELATIVE SUBDIRECT PRODUCTS

In this section, we introduce a new class of *subcentral* ideals of a CPAM. We examine some of the basic properties of subcentral ideals and characterize subdirect decompositions of a CPAM associated with complementary pairs of subcentral ideals.

Definition 6. We say that an ideal I of a CPAM $(P, \oplus, 0)$ is *subcentral* if and only if there is an ideal J such that for every $x \in P$ there is a unique decomposition $x = x_1 \oplus x_2$ with $x_1 \in I$, $x_2 \in J$. If I is subcentral ideal, then J is called the *complement* of I.

Clearly, if J is a complement of I, then J is subcentral and I is a complement of J. We denote the set of all subcentral ideals of a CPAM $\mathcal{P}$ by $SCI(P)$.

Proposition 2. Every subcentral ideal has a unique complement.

Proof. Let I be a subcentral ideal. Let J and K be complements of I. Let $x \in J$. There is a unique pair of elements x_I, x_K such that $x = x_I \oplus x_K$, where $x_I \in I$ and $x_K \in K$. Similarly, there is a unique pair x_K^I, x_K^J such that $x_K = x_K^I \oplus x_K^J$ and $x_K^I \in I$, $x_K^J \in J$. Observe that $x_K^J \in K$. It follows that $x = (x_I \oplus x_K^I) \oplus x_K^J$ is a decomposition of x into I and its complement J. Since J is a complement of I, this decomposition is unique. Evidently, $x_I \oplus x_K^I = 0$, because $x = 0 \oplus x$, $0 \in I$, $x \in J$. Thus $x = x_K^J \in K$. Analogously, $x \in K$ implies $x \in J$. ■

The above proposition allows us to define a bijective unary operation $'$ on $SCI(P)$ such that for $I \in SCI(P)$, I' is a complement of I. Note that $I'' = I$ and that $P, \{0\} \in SCI(P)$. Moreover, with every $I \in SCI(P)$ we can associate a unique mapping $\pi_I : P \mapsto I$ having the property $\pi_I(x) \oplus \pi_{I'}(x) = x$, where $\pi_I(x) \in I$ and $\pi_{I'}(x) \in I'$. Evidently, $x \in I$ iff $\pi_I(x) = x$ iff $\pi_{I'}(x) = 0$.

Proposition 3. Let I be a subcentral ideal in a CPAM $\mathcal{P}$:

(a) π_I: $P \mapsto I$ is a surjective, full homomorphism.
(b) $\forall a, b \in P$: $a \sim_I b$ iff $\pi_{I'}(a) = \pi_{I'}(b)$. Consequently, $\sim_I$ is a weak congruence.
(c) P/I is isomorphic to I'.
(d) For every $a \in P$, $\pi_I(a)$ is the greatest element of the set $\{i \in I: i \leq a\}$.

Proof. (a) π_I is surjective, because for arbitrary $x \in I$, $\pi_I(x) = x$. Let $x, y \in P$, $x \perp y$. We have

$$x \oplus y = \pi_I(x \oplus y) \oplus \pi_{I'}(x \oplus y) \tag{1}$$

$$\begin{aligned} x \oplus y &= \pi_I(x) \oplus \pi_{I'}(x) \oplus \pi_I(y) \oplus \pi_{I'}(y) \\ &= (\pi_I(x) \oplus \pi_I(y)) \oplus (\pi_{I'}(x) \oplus \pi_{I'}(y)) \end{aligned} \tag{2}$$

Both (1) and (2) are decompositions of $x \oplus y$ into I and I'. Since I is subcentral, the decomposition of $x \oplus y$ into I and I' is unique. Thus, $\pi_I(x \oplus y) = \pi_I(x) \oplus \pi_I(y)$.

Let $\pi_I(x) \perp \pi_I(y)$. In order to prove (H2), it suffices to put $x_1 = \pi_I(x)$, $y_1 = \pi_I(y)$.

(b) Assume $a \sim_I b$. Then there are $a_1, b_1 \in I$ such that $a \ominus a_1 = b \ominus b_1$. This implies $b = (a \ominus a_1) \oplus b_1$, $\pi_{I'}(b) = \pi_{I'}((a \ominus a_1) \oplus b_1) = \pi_{I'}(a \ominus a_1)$. Similarly $\pi_{I'}(a) = \pi_{I'}(b \ominus b_1)$.

Assume $\pi_{I'}(a) = \pi_{I'}(b)$. Put $a_1 = \pi_I(a)$, $b_1 = \pi_I(b)$. Then $(a \ominus a_1) = \pi_{I'}(a) = \pi_{I'}(b) = (b \ominus b_1)$.

Evidently, the above implies that $\sim_I$ satisfies (C1). The (C2) condition follows from the fact that $\pi_{I'}$ is a homomorphism.

(c) By part (b), $a \sim_I b$ iff $\pi_{I'}(a) = \pi_{I'}(b)$. Let ϕ: $P/I \mapsto I'$ be a map defined by $\phi([a]_I) = \pi_{I'}(a)$. Evidently, ϕ is well defined and bijective. Let $[a]_I \perp [b]_I$. Then, there are $a_1 \sim_I a$ and $b_1 \sim_I b$ such that $a_1 \perp b_1$. This implies that

$$\phi([a]_I) = \pi_{I'}(a) = \pi_{I'}(a_1) \perp \pi_{I'}(b_1) = \pi_{I'}(b) = \phi([b]_I)$$

Moreover,

$$\begin{aligned} \phi([a]_I \oplus [b]_I) &= \phi([a_1 \oplus b_1]_I) = \pi_{I'}(a_1 \oplus b_1) \\ &= \pi_{I'}(a_1) \oplus \pi_{I'}(b_1) = \pi_{I'}(a) \oplus \pi_{I'}(b) \end{aligned}$$

$$= \phi([a]_I) \oplus \phi([b]_I)$$

Let $\phi([a]_I) \perp \phi([b]_I)$. That means, $\pi_{I'}(a) \perp \pi_{I'}(b)$. Since $\pi_{I'}$ is full, there are $a_1, b_1 \in P$ such that $\pi_{I'}(a_i) = \pi_{I'}(a)$, $\pi_{I'}(b_1) = \pi_{I'}(b)$, and $a_1 \perp b_1$. Thus, ϕ satisfies the (H2) condition.

(d) Assume $i \in I$, $i \leq a$. There is $b \in P$ such that $a = i \oplus b$. By part (a),

$$\pi_I(a) = \pi_I(i) \oplus \pi_I(b) = i \oplus \pi_I(b)$$

Therefore, $i \leq \pi_I(a)$. ■

Definition 7. An ideal $I \in SCI(P)$ is called *central* iff $a \in I$, $b \in I'$ implies $a \perp b$.

As we will prove, the above definition of central ideal is equivalent to the definition given in ref. 5 for the class of effect algebras. Not every subcentral ideal is central, as the following example shows.

Example 2. Consider a finite CPAM given by the following table:

$\oplus$	0	a	b	c	d	e
0	0	a	b	c	d	e
a	a	·	d	·	·	·
b	b	d	·	e	·	·
c	c	·	e	·	·	·
d	d	·	·	·	·	·
e	e	·	·	·	·	·

In this CPAM, the subcentral ideals are

$$I_1 = \{0\} \qquad I_1' = P$$

$$I_2 = \{0, a\} \qquad I_2' = \{0, b, c, e\}$$

$$I_3 = \{0, c\} \qquad I_3' = \{0, a, b, d\}$$

$$I_4 = \{0, b\} \qquad I_4' = \{0, a, c\}$$

It is easy to see that only I_1, I_1', I_4, and I_4' are central.

Example 3. Let N be a set of positive integers equipped with the multiplication. Obviously, $\mathcal{N} = (N, \ldots, 1)$ is a CPAM. Since the operation on N is total, every subcentral ideal is central. There are nontrivial subcentral ideals in $\mathcal{N}$. For example, $\{2^n : n \in N \cup \{0\}\}$ is a subcentral ideal with complement $\{2n + 1 : n \in N \cup \{0\}\}$.

Definition 8. Let $\mathcal{P}_1$, $\mathcal{P}_2$ be CPAMs. Let P be a set having the following properties:

(S1) $P \subseteq P_1 \times P_2$.
(S2) P is an order ideal of $\mathcal{P}_1 \times \mathcal{P}_2$.
(S3) $\forall x_1 \in P_1: \exists x_2 \in P_2: (x_1, x_2) \in P$.
(S4) $\forall x_2 \in P_2: \exists x_1 \in P_1: (x_1, x_2) \in P$.

Then the relative subalgebra of $\mathcal{P}_1 \times \mathcal{P}_2$ associated with P is called a *relative subdirect product* of $\mathcal{P}_1$, $\mathcal{P}_2$.

It follows from Proposition 1 that a relative subdirect product of CPAMs is a CPAM.

The following two propositions establish a one-to-one correspondence between (sub)central ideals and (relative sub)direct decompositions of a CPAM. In particular, this shows that for $\mathcal{P}$ an effect algebra, our definition of central ideal of $\mathcal{P}$ is equivalent with the definition given in ref. 5.

Proposition 4. Let $\mathcal{P}$ be a relative subdirect product of $\mathcal{P}_1$ and $\mathcal{P}_2$. Then $I = \{(x_1, 0): x_1 \in P_1\}$ is a subcentral ideal of $\mathcal{P}$ and $J = \{(0, x_2): x_2 \in P_2\}$ is the complement of I. Moreover, $\mathcal{P} = \mathcal{P}_1 \times \mathcal{P}_2$ implies I is central.

Proof. First of all, let us prove $I, J \subseteq P$. Let $(x_1, 0) \in I$. By (S3), there is $x_2 \in P_2$ such that $(x_1, x_2) \in P$. Now, $(x_1, x_2) = (x_1, 0) \oplus (0, x_2)$ in $\mathcal{P}_1 \times \mathcal{P}_2$. It follows that $(x_1, 0) \in P$, since P is an order ideal of $\mathcal{P}_1 \times \mathcal{P}_2$. One can prove $J \subseteq P$ in a similar way.

It is easy to check that both I and J are ideals of P. It remains to prove that I is subcentral ideal.

Let $(x_1, x_2) \in P$. Then $(x_1, x_2) = (x_1, 0) \oplus_P (0, x_2)$, where $(x_1, 0) \in I$ and $(0, x_2) \in J$. This decomposition is unique. Indeed, let $(x_1, x_2) = (y_1, y_2) \oplus_P (z_1, z_2)$, $(y_1, y_2) \in I$, $(z_1, z_2) \in J$. Clearly, $y_2 = z_1 = 0$. Now, $(x_1, x_2) = (y_1, 0) \oplus_P (0, z_2)$, which implies $x_1 = y_1$ and $x_2 = z_2$.

Suppose $\mathcal{P} = \mathcal{P}_1 \times \mathcal{P}_2$. Obviously, for all $x_1 \in P_1$ and $x_2 \in P_2$, $(x_1, 0) \perp (0, x_2)$ holds. Thus I is central. ■

Proposition 5. Let I be a subcentral ideal of a CPAM $\mathcal{P}$. Then $\mathcal{P}$ is isomorphic to a relative subdirect product of I and I'. Moreover, if I is central, then P is isomorphic to $I \times I'$.

Proof. We denote $Q = I \times I'$ and $\oplus_Q$ the operation on Q. Let $A = \{(x_1, x_2): x_1 \in I, x_2 \in I', x_1 \perp x_2\}$, and let $\mathcal{A} = (A, \oplus_A, 0)$ be a relative subalgebra of Q associated with A.

(S1) holds by definition of A. For a proof of (S2), let $(x_1, x_2) \in A$, $(y_1, y_2) \leq_Q (x_1, x_2)$. Evidently, $y_1 \leq_P x_1$ and $y_2 \leq_P x_2$. Thus, $x_1 \perp x_2$ implies $y_1 \perp y_2$, i.e., $(y_1, y_2) \in A$. Since $x_1 \in I$ implies $(x_1, 0) \in A$, (S3) holds. The proof of (S4) is similar. Thus, $\mathcal{A}$ is a relative subdirect product of I and I'. It remains to prove that $\mathcal{A}$ is isomorphic to $\mathcal{P}$.

Let $\varphi: P \mapsto A$ be defined by $\varphi(x) = (\pi_I(x), \pi_{I'}(x))$. We will prove that φ is an isomorphism. For a proof of injectivity, assume $\varphi(x) = \varphi(y)$. That means $(\pi_I(x), \pi_{I'}(x)) = (\pi_I(y), \pi_{I'}(y))$. It follows that $x = \pi_I(x) \oplus \pi_{I'}(x) = \pi_I(y) \oplus \pi_{I'}(y) = y$. To prove surjectivity, observe that for $(x_1, x_2) \in A$, $\varphi(x_1 \oplus x_2) = (x_1, x_2)$. (H1) follows from the fact that π_I and $\pi_{I'}$ are homomorphisms.

Let $\varphi(x) = (\pi_I(x), \pi_{I'}(x)) \perp_A \varphi(y) = (\pi_I(y), \pi_{I'}(y))$. Since $\varphi(x)$, $\varphi(y)$ are orthogonal in $\mathcal{A}$, their sum must be an element of A: $(\pi_I(x) \oplus \pi_I(y), \pi_{I'}(x) \oplus \pi_{I'}(y)) \in A$. Therefore, $(\pi_I(x) \oplus \pi_I(y)) \perp_P (\pi_{I'}(x) \oplus \pi_{I'}(y))$, which implies $x = \pi_I(x) \oplus \pi_{I'}(x) \perp_P \pi_I(y) \oplus \pi_{I'}(y) = y$.

Suppose I is central. Then $A = Q$. The rest is obvious. ■

4. SUBCENTRAL IDEALS IN SOME IDEAL LATTICES

In this section, we investigate the role of subcentral ideals in the lattice of R_1-ideals, recently introduced in ref. 1. We prove that $(SCI(P), \subseteq)$ is a Boolean lattice which is the center of the lattice of R_1-ideals. Moreover, every subcentral ideal is a central element of the lattice $I(P)$ of all ideals. A subcentral ideal is central iff it is a Riesz ideal.

Definition 9. Let $\mathcal{P}$ be a CPAM, $I \in I(P)$. I is called a R_1-*ideal* iff for every $i \in I$ and $a, b \in P$ such that $a \perp b$, $i \le a \oplus b$, there are $i_1, i_2 \in I$ such that $i_1 \le a$, $i_2 \le b$, $i \le i_1 \oplus i_2$. An R_1 ideal I is called a *Riesz ideal* iff for every $i \in I$, $a, b \in P$, $i \le a$, and $a \ominus i \perp b$ there exists $j \in I$ such that $j \le b$ and $a \perp b \ominus j$.

We denote $R_1\, I(P)$ the set of all R_1-ideals of a CPAM $\mathcal{P}$. The following proposition summarizes some of the results from ref. 1.

Proposition 6. Let $\mathcal{P}$ be a CPAM.

- (a) Let $I \in R_1I(P)$, $J \in I(P)$. $I \vee J = I \oplus J = \{i \oplus j : i \in I, j \in J, i \perp j\}$.
- (b) $R_1I(P)$ is a complete distributive sublattice of $I(P)$.
- (c) If $(P, \le)$ is upper-directed, then every R_1-ideal is a Riesz ideal.
- (d) If I is an R_1-ideal, then $\sim_I$ satisfies (C1), (C2), and (C5). Consequently, P/I is a PAM.

Recall that for a bounded lattice L, an element $z \in L$ is called *central* iff there is a $z' \in L$ such that L is isomorphic to $[0, z] \times [0, z']$. The set of all central elements of a lattice L [denoted by $C(L)$] forms a Boolean sublattice of L.

Proposition 7. Let $\mathcal{P}$ be a CPAM. Then $SCI(P) = C(R_1I(P))$.

Proof. Let $I \in SCI(P)$, $i \in I$, $i \le a \oplus b$. Then $\pi_I(i) \le \pi_I(a) \oplus \pi_I(b)$, $\pi_I(a) \le a$, $\pi_I(b) \le b$, showing that $I \in R_1I(P)$.

By ref. 9, Theorem 4.13, an element of a distributive lattice is central iff it has a complement. For $I \in SCI(P)$, $I \vee I' = I \oplus I' = P$ and $I \wedge I' = \{0\}$. Therefore, $SCI(P) \subseteq C(R_1I(P))$.

For the opposite direction, let $I \in C(R_1I(P))$. Then I has a complement I', i.e., $P = I \oplus I'$, $I \wedge I' = \{0\}$. Evidently, every $x \in P$ has a decomposition $x = x_I \oplus x_{I'}$, $x_I \in I$, $x_{I'} \in I'$. It remains to prove that this decomposition is unique.

Let us denote $p_I(x) = \{x_I, x_I \in I, x_I \leq x, x \ominus x_I \in I'\}$. A decomposition of x into I and I' exists; therefore $p_I(x)$ is not empty. Let $i \in p_I(a \oplus b)$. Then $i \leq a \oplus b$ and since I is an R_1-ideal, there are $i_1, i_2 \in I$ such that $i \leq i_1 \oplus i_2$, $i_1 \leq a$, $i_2 \leq b$, $i \leq i_1 \oplus i_2$. Denote $c = (i_1 \oplus i_2) \ominus i$, $a_1 = a \ominus i_1$, $b_1 = b \ominus i_2$. Note that $c \in I$ and $a \oplus b = i_1 \oplus a_1 \oplus i_2 \oplus b_1 = i \oplus c \oplus a_1 \oplus b_1$. The later equation implies $(a \oplus b) \ominus i = c \oplus a_1 \oplus b_1 \in I'$ because $i \in p_I(a \oplus b)$. It follows, $c \in I \cap I' = \{0\}$, $a_1 \in I'$, $b_1 \in I'$. Thus, $i_1 \in p_I(a)$, $i_2 \in p_I(b)$, $i = i_1 \oplus i_2$, $p_I(a \oplus b) \subseteq p_I(a) \oplus p_I(b)$.

Now let $x = x_I \oplus x_{I'}$, $x_I \in I$, $x_{I'} \in I'$. $p_I(x) \subseteq p_I(x_I) \oplus p_I(x_{I'})$. Observe that $p_I(x_{I'}) \subseteq I \cap I' = \{0\}$ and $p_I(x_I) = \{x_I\}$. This, together with $p_I(x) \neq \emptyset$, implies $p_I(x) = \{x_I\}$. Similarly, $p_I^{I'}(x) = \{x_{I'}\}$. The decomposition of x into I, I' exists and is unique, $I \in SCI(P)$. ■

Corollary 1. $SCI(P)$ is a Boolean sublattice of $I(P)$.

Proposition 8. Let $I, J \in SCI(P)$, $x \in P$.
(a) $\pi_{I \wedge J}(x) = \pi_J(\pi_I(x))$
(b) $\pi_{I \vee J}(x) = \pi_J(\pi_{I'}(x)) \oplus \pi_I(x)$

Proof. (a) $(I \wedge J)' = I' \vee J'$, since $SCI(P)$ is a Boolean algebra. Now, $x = \pi_I(x) \oplus \pi_{I'}(x) = \pi_J(\pi_I(x)) \oplus \pi_{J'}(\pi_I(x)) \oplus \pi_{I'}(x)$ is a decomposition of x into $I \wedge J \ni \pi_J(\pi_I(x))$ and $I' \vee J' \ni \pi_{J'}(\pi_I(x)) \oplus \pi_{I'}(x)$. This implies $\pi_{I \wedge J}(x) = \pi_I(\pi_J(x))$.
(b) Similarly. ■

Corollary 2. Let $I, J \in SCI(P)$. Then $\pi_I \circ \pi_J = \pi_J \circ \pi_I$.

Proof. Let $x \in P$. Then $\pi_I(\pi_J(x)) = \pi_{I \wedge J}(x) = \pi_{J \wedge I}(x) = \pi_J(\pi_I(x))$. ■

Proposition 9. Let $\mathcal{P}$ be a CPAM. Then $SCI(P) \subseteq C(I(P))$.

Proof. Let $J \in I(P)$, $I \in SCI(P)$. According to ref. 9, Theorem 4.13, it is sufficient to prove

$$J = (J \wedge I) \vee (J \wedge I') = (J \vee I) \wedge (J \vee I')$$

(1) $J = (J \wedge I) \vee (J \wedge I')$.
By Proposition 3 of ref. 1, $I, I' \in R_1I(P)$ implies that $(J \wedge I) \vee (J \wedge I') = J \wedge (I \vee I') = J$.

(2) $J = (J \vee I) \wedge (J \vee I')$.
$J \subseteq (J \vee I) \wedge (J \vee I')$ is trivial. By ref. 1, Proposition 2, $I \in R_1 I(P)$ and $K \in I(P)$ implies $I \vee K = I \oplus K$. Consequently,

$$(J \vee I) \wedge (J \vee I') = (J \oplus I) \wedge (J \oplus I')$$

Let $a \in (J \oplus I) \wedge (J \oplus I')$. We have $a \in J \oplus I$ and so there are $j \in J$ and $i \in I$ such that $a = j \oplus i$. Then

$$\pi_{I'}(a) = \pi_{I'}(j) \oplus \pi_{I'}(i) = \pi_{I'}(j) \oplus 0 \leq j \in J$$

so $\pi_{I'}(a) = \pi_{I'}(j) \in J$. Similarly, $a \in J \oplus I'$ implies $\pi_I(a) \in J$. Since $\pi_I(a) \in J$ and $\pi_{I'}(a) \in J$, $a = \pi_I(a) \oplus \pi_{I'}(a) \in J$. ■

The following proposition gives a characterization of central ideals.

Proposition 10. Let $\mathcal{P}$ be a CPAM, let $I \in SCI(P)$. I is central iff I is a Riesz ideal.

Proof. Let I be a Riesz, subcentral ideal. Let $a \in I$, $b \in I'$. Consider the definition of a Riesz ideal. Put $i = a$. Evidently $i \leq a$ and $a \ominus i = 0 \perp b$. This implies that there is $j \in I$, $j \leq b$, such that $b \ominus j \perp a$. Now, $j \leq b \in I'$ implies that $j \in I'$. Consequently, $j \in I \wedge I' = \{0\}$. Therefore $j = 0$ and $b \ominus j = b \perp a$.

The opposite implication follows from Example 7 in ref. 6, using the results of the previous section. However, we present a direct proof here.

Let I be a central ideal. Assume $a \geq i \in I$, $a \ominus i \perp b$. Since I is central, $\pi_I(a) \perp \pi_{I'}((a \ominus i) \oplus b)$. We have

$$\begin{aligned}\pi_I(a) \oplus \pi_{I'}((a \ominus i) \oplus b) &= \pi_I(a) \oplus \pi_{I'}(a \ominus i) \oplus \pi_{I'}(b)\\ &= \pi_I(a) \oplus (\pi_{I'}(a) \ominus \pi_{I'}(i)) \oplus \pi_{I'}(b)\\ &= \pi_I(a) \oplus \pi_{I'}(a) \oplus \pi_{I'}(b) = a \oplus \pi_{I'}(b)\end{aligned}$$

Thus, $a \perp \pi_{I'}(b) = b \ominus \pi_I(b)$ and, putting $j = \pi_I(b)$, we see that $a \perp b \ominus j$. ■

Corollary 3. Let $\mathcal{P}$ be an upper-directed CPAM (in particular, an effect algebra). Then every subcentral ideal is central.

Proof. By ref. 1, Proposition 9, in an upper-directed CPAM every R_1-ideal is Riesz. ■

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REFERENCES

1. Chevalier, G., and Pulmannová, S. Some ideal lattices in Partial Abelian Monoids, preprint.
2. Foulis, D., and Bennett, M. K., Effect algebras and unsharp quantum logics, *Found. Phys.* **24** (1994) 1331–1352.
3. Giuntini, R., and Greuling, H., Toward a formal language for unsharp properties, *Found. Phys.* **19** (1994) 769–780.
4. Grätzer, G. *Universal Algebra*, 2nd ed., Springer-Verlag, Berlin (1979), pp. 80–81.
5. Greechie, R., Foulis, D., and Pulmannová, S. The center of an effect algebra, *Order* **12** (1995) 91–106.
6. Gudder, S., and Pulmannová, S., Quotients of partial abelian monoids, *Algebra Univers.* **38** (1998) 395–421.
7. Kalmbach, G., and Riečanová, Z. An axiomatization for abelian relative inverses, *Dem. Math.* **27** (1994) 684–699.
8. Kôpka, F., and Chovanec, F., D-posets, *Math. Slovaca* **44** (1994) 21–34.
9. Maeda, F., and Maeda, S., *Theory of Symmetric Lattices*, Springer-Verlag, Berlin, (1970), pp. 18–19.
10. Pulmannová, S., Congruences in partial abelian semigroups, *Algebra Univers.* **37** (1997) 119–140.
11. Wilce, W., Partial abelian semigroups, *Int. J. Theor. Phys.* **34** (1995) 1807–1812.